An Algebraic Approach to Voting Theory

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Voting Theory Basics

Our society is very familiar with the process of voting and pooling preferences. Unfortunately, it is not always easy to make a single choice when given the preferences of many voters, let alone decide how to ask for their preferences.

Big Idea: Which voting system is best?

Two Methods of Voting:

A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

Full rankings: Above are all possible full rankings of three items. Full rankings require strict preferences.

A	A	A	B	B
B	C	D	A	C
CD	BD	BC	CD	AD

Partial rankings: Above are some possible ways to rank 2 items from 4. Partial rankings allow for ties, and are given by their **shape** λ: For above, $\lambda = (1,1,2)$.

Two Methods of Tallying Votes:

Positional method: each candidate receives points according to where they are placed in each ranking. Each method has a **weighting vector** $\mathbf{w}$ which represents the number of points given to each candidate. Example: the **Borda Count** has $\mathbf{w} = [\frac{n-1}{n-1}, \frac{n-2}{n-1}, \ldots, 0]$, where n = # of candidates.

Pairwise method: compare candidates pairwise, such that $A > B$ if A is ranked higher than B more times than B is over A. Also known as **Condorcet's method**.

An Example:

Say 24 votes have been cast for full rankings as follows:

	A	A	B	B	C	C
Ranking:	B	C	A	C	A	B
	C	B	C	A	B	A
# of votes:	3	8	8	0	0	5

We express this data as a **profile** (a vector representing the number of votes cast for each ranking), $\mathbf{p} = (3,8,8,0,0,5)$.

- **Positional outcome ($\mathbf{w} = [1, 1/2, 0]$):** A wins
  A gets 15 points, B gets 12 points, C gets 9 points.
- **Pairwise outcome:**
  B beats A 13 : 11, A beats C 19 : 5, C beats B 13 : 11.

The pairwise rankings are not transitive, nor do they agree with the positional ranking. These are called **paradoxes**.

Previous Contributions

Arrow's Theorem:

Addressing the search for the ideal method, Arrow proved that the following four criteria for an election procedure are inconsistent:
Universality: the procedure should provide a full ranking for all possible sets of data.
Independence of Irrelevant Alternatives: any ranking of a subset of candidates will be unaffected by changes in rankings of other candidates.
Citizen's Sovereignty: all possible outcomes are achievable.
Non-dictatorship: outcome dictated by more than one vote.

Saari Breaks Down the Profile Space:

Donald Saari compared pairwise and positional methods for *fully ranked* data, and showed that **the Borda Count is the unique positional method to minimize conflict** between these. Saari also constructed bases for the profile spaces, and broke them into the following four subspaces, according to how they behaved under each procedure:
Kernel: profiles which tie under any tally method.
Basic: profiles for which all positn'l & pairwise tallies agree.
Condorcet: profiles only influencing pairwise outcomes.
Reversal: profiles only influencing positional outcomes.

For example, the following table outlines Saari's basis for a full ranking of three candidates. The first row corresponds to the basis itself, while the second and third rows show the images of the basis vectors under the positional ($\mathbf{w} = [1, t, 0]$) and pairwise methods, respectively.

	Kernel	Basic $\mathbf{b}_A$	Basic $\mathbf{b}_B$	Basic $\mathbf{b}_C$	Cond.	Reversal $\mathbf{r}_A$	Reversal $\mathbf{r}_B$	Reversal $\mathbf{r}_C$
Basis:								
ABC	1	1	0	−1	1	1	−2	1
ACB	1	1	−1	0	−1	1	1	−2
BAC	1	0	1	−1	−1	−2	1	1
BCA	1	−1	1	0	1	1	1	−2
CAB	1	0	−1	1	1	−2	1	1
CBA	1	−1	0	1	−1	1	−2	1
Positional outcome:								
A	$2+2t$	2	−1	−1	0	$2-4t$	$-1+2t$	$-1+2t$
B	$2+2t$	−1	2	−1	0	$-1+2t$	$2-4t$	$2-4t$
C	$2+2t$	−1	−1	2	0	$-1+2t$	$-1+2t$	$-1+2t$
Pairwise outcome:								
AB	3	2	−2	0	1	0	0	0
BA	3	−2	2	0	−1	0	0	0
AC	3	2	0	−2	−1	0	0	0
CA	3	−2	0	2	1	0	0	0
BC	3	0	2	−2	1	0	0	0
CB	3	0	−2	2	−1	0	0	0

Table 1: Saari's basis for the profile space on tree alternatives, and each basis vector's image under the positional and pairwise maps.

Algebraic Tools

Representing Tally Methods as Linear Maps:

Ex. Positional method: 3 candidates, $\mathbf{w} = [1, t, 0]$.

$$T_{\mathbf{w}} = \begin{pmatrix} 1 & 1 & t & 0 & t & 0 \\ t & 0 & 1 & 1 & 0 & t \\ 0 & t & 0 & t & 1 & 1 \end{pmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$

The columns correspond to rankings, and the rows to candidates. The highlighted column shows that the ranking $A > B > C$ will give A 1 point, B t points, and C no points.

Major Tools from Representation Theory:

$\mathbb{Q}S_n \equiv$ formal sums of permutations on n elements, with coefficients from $\mathbb{Q}$.
$X^\lambda \equiv$ set of all λ-shaped rankings.
$M^\lambda \equiv \mathbb{Q}X^\lambda$, the profile space for λ-shaped rankings. We treat M^λ as a $\mathbb{Q}S_n$-module.

Thm.: All pairwise and positional maps are $\mathbb{Q}S_n$-module homomorphisms.
Thm.: *(Maschke)* If M is a nontrivial $\mathbb{Q}S_n$-module then it can be expressed as the direct sum of irreducible submodules.
Thm.: *(Schur)* If M and N are two irreducible $\mathbb{Q}S_n$-modules, then any nonzero $\mathbb{Q}S_n$-module homomorphism from M to N is an isomorphism.

An Example:

Say M and N are $\mathbb{Q}S_n$-modules, $\varphi : M \to N$ is a $\mathbb{Q}S_n$-module homomorphism, and

$$M \cong W \oplus W \oplus W \oplus Y \oplus Z$$
$$N \cong W \oplus W \oplus Y \oplus Y.$$

Then $\varphi(M) \subseteq W \oplus W \oplus Y$. So, for all $z \in Z$, $\varphi(z) = 0$.

Big Idea: We can decompose profile spaces to decipher what happens under different maps.

Some Useful Decompositions:

Full rankings of three items:
$M^{(1,1,1)} \cong S^{(3)} \oplus 2S^{(2,1)} \oplus S^{(1,1,1)}$

Full rankings of n items:
$M^{(1,\ldots,1)} \cong S^{(n)} \oplus (n-1)S^{(n-1,1)} \oplus 2(n-3)S^{(n-2,2)} \oplus \cdots$

Rankings of 3 items from 5:
$M^{(n-3,1,1,1)} \cong S^{(n)} \oplus 3S^{(n-1,1)} \oplus 3S^{(n-2,2)} \oplus 3S^{(n-2,1,1)} \oplus \cdots$

Positional outcomes:
$M^{(n-1,1)} \cong S^{(n)} \oplus S^{(n-1,1)}$

Pairwise outcomes:
$M^{(n-2,1,1)} \cong S^{(n)} \oplus 2S^{(n-1,1)} \oplus S^{(n-2,2)} \oplus S^{(n-2,1,1)}$

Results

For Fully Ranked Data:

Borda Count still wins: The Borda Count is the unique positional method for which the following diagram commutes for some ς. It is the only map for which any profile in the kernel of P (i.e., does not affect the pairwise outcome) is also in the kernel of $T_{\mathbf{w}}$ (does not affect the positional outcome).

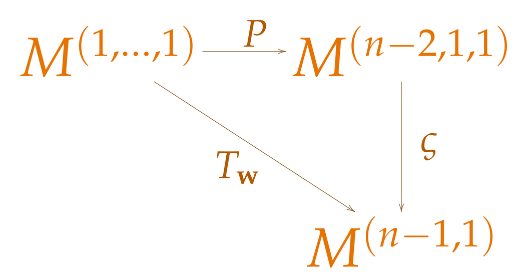

Figure 1: Diagram of maps between fully ranked profile and tally spaces. P is the pairs map, $T_{\mathbf{w}}$ is the positional map with weighting vector $\mathbf{w}$, and ς is an arbitrary $\mathbb{Q}S_n$-modl. homomorphism from the pairs space to the positional space. See section on "Useful Decompositions."

For Partially Ranked Data:

In some circumstances, we will have more freedom in our choice of $\mathbf{w}$. However, if we give each item in a tie 1/2 point for the pairs map, we are again restricted to a unique positional method. In either case, we want positional weights which allow the following diagram to commute:

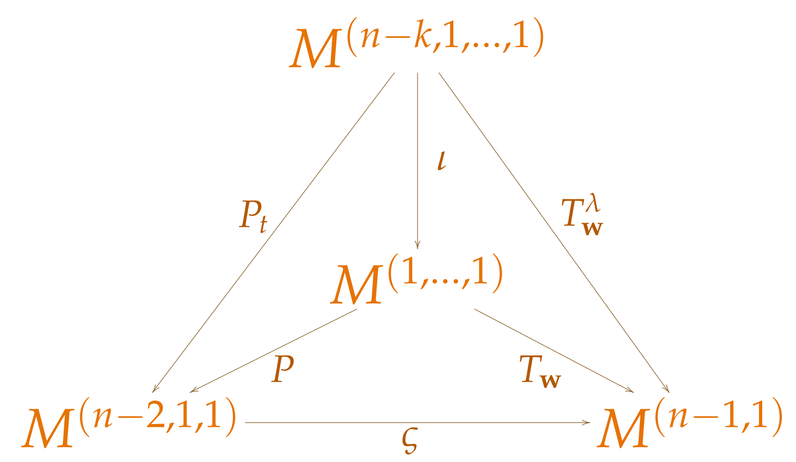

Figure 2: Complete diagram of maps between profile and tally spaces. P_t is the pairs map where ties receive t points, $T^\lambda_{\mathbf{w}}$ is the positional map for partial rankings of shape λ, and ι is an $\mathbb{Q}S_n$-module hom. mapping partially ranked profiles into fully ranked profiles. See Fig. 1 above.

References

[1] David S. Dummit and Richard M. Foote. *Abstract Algebra.* Upper Saddle River, N.J., Prentice Hall, 1999.

[2] Donald G. Saari. Explaining all three-alternative voting outcomes. *J. Econom. Theory*, 87(2):313355, 1999.

[3] Bruce Sagan. *The Symmetric Group* Springer-Verlag New York, Inc., 2001.